

Fundamentals of traffic flow

Dirk Helbing

II. Institute of Theoretical Physics, University of Stuttgart, Pfaffenwaldring 57/III, 70550 Stuttgart, Germany

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From single-vehicle data a number of empirical results concerning the density dependence of the velocity distribution and its moments, as well as the characteristics of their temporal fluctuations, have been determined. These are utilized for the specification of some fundamental relations of traffic flow and compared with existing traffic theories. [S1063-651X(97)05402-0]

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For prosperity in industrialized countries, efficient traffic systems are indispensable. However, due to an overall increase of mobility and transportation during the last years, the capacity of the road infrastructure has been reached. Some cities like Los Angeles and San Francisco already suffer from daily traffic collapses and their environmental consequences. About 20% more fuel consumption and air pollution is caused by impeded traffic and stop-and-go traffic.

For the above-mentioned reasons, several models for free-way traffic have been proposed, microscopic and macroscopic ones (for an overview cf. Ref. [1]). These are used for developing traffic optimization measures such as on-ramp control, variable speed limits, or rerouting systems [1]. For such purposes, the best models must be selected and calibrated to empirical traffic relations. However, some relations are difficult to obtain, and the lack of available empirical data has caused some stagnation in traffic modeling.

Further advances will require a close interplay between theoretical and empirical investigations [2]. On the one hand, empirical findings are necessary to test and calibrate the various traffic models. On the other hand, some hardly measurable quantities and relations can be reconstructed by means of theoretical relations.

Therefore, a number of fundamental traffic relations will be presented in the following. Until now, little is known about the velocity distribution of vehicles, its variance or skewness. A similar thing holds for the functional form of the velocity-density relation or the variance-density relation at high densities. Empirical results have also been missing for the fluctuation characteristics of the density or average velocity. These gaps will be closed in the following. Although the data vary in detail from one freeway stretch to another, the essential conclusions are expected to be universal.

In a recent paper [3], it has been shown that the traffic dynamics in neighboring lanes is strongly correlated. Therefore, it is possible to treat the total freeway cross section in an overall way. Consequently, we will only discuss the properties of the *lane averages* of macroscopic traffic quantities. The empirical relations have been evaluated from single vehicle data of the Dutch two-lane freeway A9 between Haarlem and Amsterdam (for a sketch, cf. Fig. 1 in Ref. [3]). These data were detected by induction loops at discrete places x of the roadway and include the passage times $t_\alpha(x)$, velocities $v_\alpha(x)$, and lengths $l_\alpha(x)$ of the single vehicles α . Consequently, it was possible to calculate the number $N(x,t)$

of vehicles which passed the cross section at location x during a time interval $[t-T/2, t+T/2]$, the *traffic flow*

$$Q(x,t) := N(x,t)/T, \tag{1}$$

and the macroscopic *velocity moments*

$$\langle v^k \rangle := \frac{1}{N(x,t)} \sum_{t-T/2 \leq t_\alpha(x) < t+T/2} [v_\alpha(x)]^k. \tag{2}$$

Small values of T are connected with large statistical variations of the data, but large values can cause biased results for $k \geq 2$ [3]. Values between 0.5 and 2 min seem to be the best compromise [1]. The *vehicle densities* $\rho(x,t)$ were calculated via the theoretical flow formula

$$Q(x,t) = \rho(x,t)V(x,t). \tag{3}$$

Other evaluation methods [4] are discussed in Ref. [1].

We start with the discussion of the grouped empirical velocity distribution $P(v;x,t)$, which was obtained in the usual way,

$$P(v_l;x,t) := \frac{n(x,v_l,t)}{N(x,t)}. \tag{4}$$

Here, $n(x,v_l,t)$ denotes the number of vehicles that pass the cross section at x between times $t-T/2$ and $t+T/2$ with a velocity $v \in [v_l - \Delta/2, v_l + \Delta/2)$. The class interval length was chosen $\Delta = 5$ km/h.

In *theoretical* investigations, the velocity distribution $P(v;x,t)$ has mostly been assumed to have the Gaussian form [5-7]

$$P_G(v;x,t) := \frac{1}{\sqrt{2\pi}\Theta(x,t)} \exp\left(-\frac{[v-V(x,t)]^2}{2\Theta(x,t)}\right). \tag{5}$$

Here, $V(x,t) := \langle v \rangle$ denotes the *average velocity* and $\Theta(x,t) := \langle [v-V(x,t)]^2 \rangle$ denotes the *velocity variance*. Assumption (5) has been made for two reasons: First, it allows us to derive approximate fluid-dynamic traffic equations from a gas-kinetic level of description [5-7]. Second, analytical results for the velocity distribution are not yet available, even for the stationary and spatially homogeneous case. Therefore, the question is whether the Gaussian approximation is justified or not. Figure 1 gives a positive answer, at

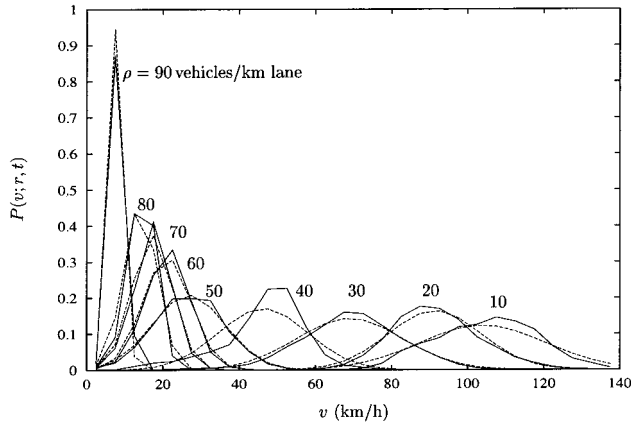


FIG. 1. Comparison of empirical velocity distributions at different densities (—) with frequency polygons of grouped Gaussian velocity distributions with the same mean value and variance (---). A significant deviation of the empirical relations from the respective discrete Gaussian approximations is only found at a density of $\rho=40$ vehicles/km lane, where the temporal averages over $T=2$ min may have been too long due to rapid stop-and-go waves [3] (cf. the mysterious “knee” at $\rho\approx 40$ veh/km in Fig. 5).

least for the average velocity distribution at small and medium densities. In particular, bimodal distributions are not observed [8].

An investigation of the *temporal evolution* of the velocity distribution is difficult due to the large statistical fluctuations (which come from the fact that only a few vehicles per velocity class pass the observed freeway cross section during the short time period T). Therefore, we will study a macroscopic (aggregated) quantity instead, namely, the temporal variation of the *skewness*

$$\gamma(x, t) := \frac{\langle [v - V(x, t)]^3 \rangle}{[\Theta(x, t)]^{3/2}} = \frac{\langle v^3 \rangle - 3\langle v \rangle \langle v^2 \rangle + 2\langle v \rangle^3}{[\Theta(x, t)]^{3/2}}. \quad (6)$$

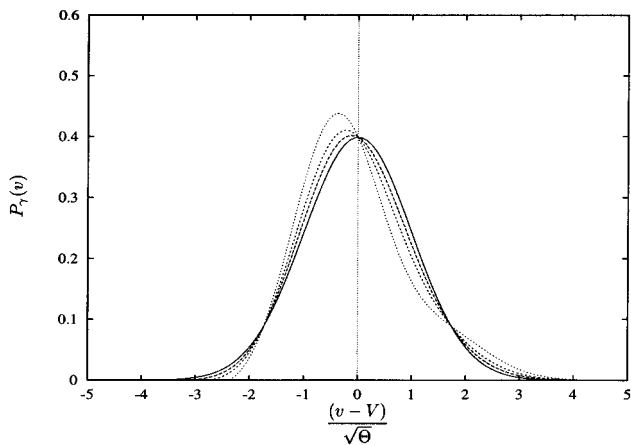


FIG. 2. Velocity distributions $P_\gamma(v) := \{1 - \gamma[3(v-V)/\Theta^{1/2} - (v-V)^3/\Theta^{3/2}]/6\}P_G(v)$ with the same average velocity V and variance Θ , but different values of the skewness γ (—: $\gamma=0$; ---: $\gamma=1/2$; ···: $\gamma=1$; ····: $\gamma=2$). Obviously, a skewness of $|\gamma|\leq 0.5$ only leads to minor changes compared to the Gaussian distribution (—).

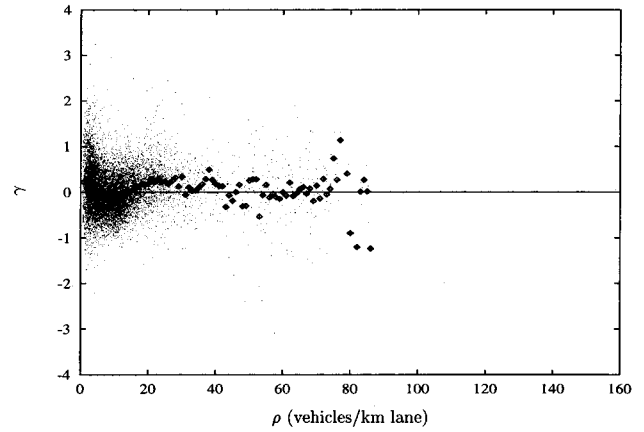


FIG. 3. Density dependence of the skewness γ (·: 1-min data; \diamond : respective mean values). The large variation of the 1-min data at low densities is due to the small number of vehicles that pass a cross section during the time interval $T=1$ min, whereas the large variation of their mean values at high densities comes from the few 1-min data, over which it could be averaged. The 1-min data of the skewness scatter around the zero line (—) and mostly lie between -0.5 and 0.5 .

This can be interpreted as a dimensionless measure of asymmetry (cf. Fig. 2). Figure 3 shows that the skewness mainly varies between -0.5 and 0.5 . The deviation from zero is neither systematic nor significant, so that the skewness is normally negligible. This indicates that even the time-dependent velocity distribution is approximately Gaussian shaped [9].

Now we will investigate how the average velocity V and the variance Θ depend on the vehicle density ρ (cf. Figs. 4 and 5). The problem is that the data for high vehicle densities are missing. However, for computer simulations of traffic dynamics, the corresponding functional relations need to be specified. This can be done by means of theoretical results. For the average velocity and variance on freeways with speed limits, recent gas-kinetic traffic models [6] imply the following implicit *equilibrium relations* (indicated by a subscript e), if the skewness is neglected (cf. Fig. 3):

$$V_e(\rho) = V_0 - \frac{\tau(\rho)[1 - p(\rho)]\rho\Theta_e(\rho)}{1 - \rho/\rho_{\max} - \rho T_r V_e(\rho)}, \quad (7)$$

and

$$\Theta_e(\rho) = A(\rho)[V_e(\rho)^2 + \Theta_e(\rho)], \quad \text{i.e.,} \\ \Theta_e(\rho) = \frac{A(\rho)V_e(\rho)^2}{1 - A(\rho)}. \quad (8)$$

Here, V_0 denotes the *average desired speed* (or free speed), $\tau(\rho)$ is the effective density-dependent *relaxation time* of acceleration maneuvers, and $p(\rho)$ means the *probability of immediate overtaking*. Moreover, ρ_{\max} denotes the *maximum vehicle density*, T_r the *reaction time*, and $A(\rho)$ with $0 \leq A(\rho) \leq 1$ the *relative individual velocity fluctuation* during the time interval $\tau(\rho)$ [1,6].

According to relation (8), the equilibrium variance vanishes when the average velocity becomes zero. This consistency condition is not met by every traffic model (cf. Ref.

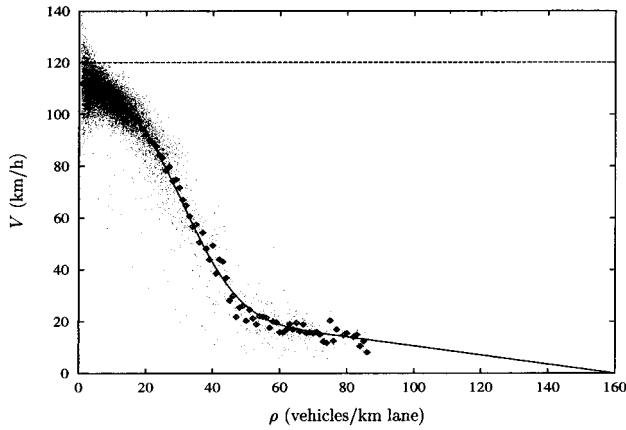


FIG. 4. Relation between average velocity and density [\cdot : 1-min data; \diamond : respective mean values; —: fit function for the equilibrium relation $V_e(\rho)$]. The speed limit is 120 km/h (—).

[10]). In addition, we expect that the average velocity vanishes at the maximum vehicle density ρ_{\max} . Therefore, in the limit $\rho \rightarrow \rho_{\max}$, we must have the proportionality relation

$$\tau(\rho)[1-p(\rho)]\rho \frac{A(\rho)V_e(\rho)^2}{1-A(\rho)} \propto 1 - \frac{\rho}{\rho_{\max}} - \rho T_r V_e(\rho), \quad (9)$$

the proportionality factor being V_0 . Whereas the overtaking probability $p(\rho)$ is expected to vanish for $\rho \rightarrow \rho_{\max}$, the relaxation time $\tau(\rho)$ and the fluctuation parameter $A(\rho)$ are assumed to remain finite [11]. Therefore, the *ansatz* $V_e(\rho) \propto (1 - \rho/\rho_{\max})^\beta$ leads to $\beta=1$ and

$$V_e(\rho) = \frac{\rho_{\max} - \rho}{T_r(\rho_{\max})^2} \quad \text{for } \rho \approx \rho_{\max}. \quad (10)$$

This is a very interesting discovery, since many researchers believed that the average velocity approaches the ρ axis horizontally. In addition, we find that $\Theta_e(\rho) \propto (1 - \rho/\rho_{\max})^2$ for $\rho \rightarrow \rho_{\max}$.

Our remaining task is to specify the parameters ρ_{\max} and T_r . From other measurements, it is known that ρ_{\max} lies between 160 and 180 vehicles per kilometer and lane [12].

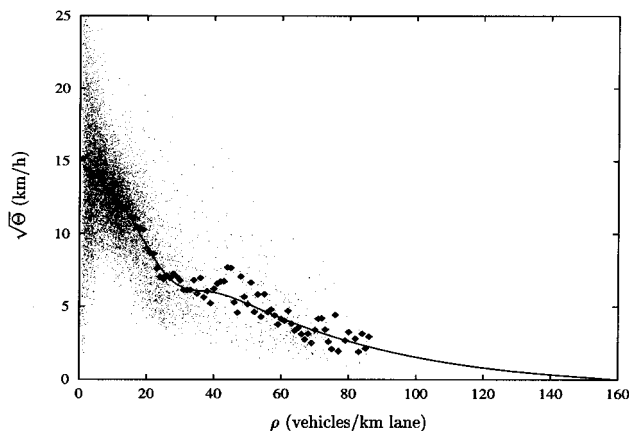


FIG. 5. Density dependence of the standard deviation $\sqrt{\Theta}$ of the vehicle velocities [\cdot : 1-min data; \diamond : respective mean values; —: fit function for the equilibrium relation $\sqrt{\Theta_e(\rho)}$].

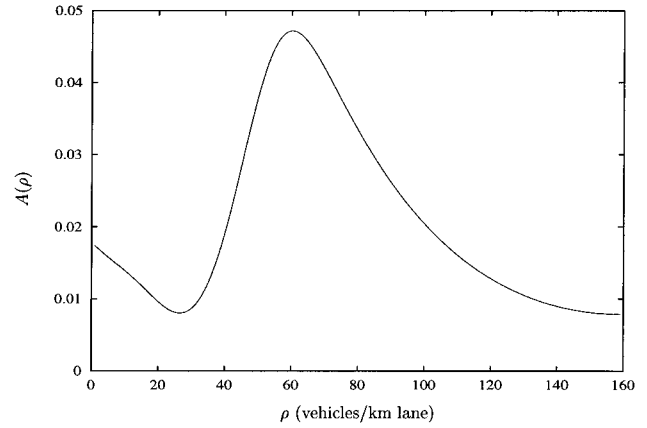


FIG. 6. Density dependence of the fluctuation strength $A(\rho)$, which is a measure of the relative velocity variation during a time interval $\tau(\rho)$. Its maximum at medium densities indicates that velocity fluctuations are particularly large in the region of unstable traffic flow.

The reaction time T_r for expected events is at least 0.7 s [13]. A good fit of the data results for

$$\rho_{\max} = 160 \text{ vehicles/km lane}, \quad T_r = 0.8 \text{ s} \quad (11)$$

(cf. Fig. 4). In addition, we can conclude from Eq. (7) that the velocity-density relation $V_e(\rho)$ of a multilane freeway should start horizontally, since the probability of overtaking $p(\rho)$ should approach the value 1 at very small densities $\rho \approx 0$.

However, it is not only possible to reconstruct the functional forms of the velocity-density relation $V_e(\rho)$ and the variance-density relation $\Theta_e(\rho)$. From these, we can also determine the dependence of the model functions $A(\rho)$ and $\tau(\rho)[1-p(\rho)]$ by means of the theoretical relations (7) and (8). The result for the diffusion strength $A(\rho)$ is depicted in Fig. 6.

Finally, we will investigate the temporal fluctuations of the empirical vehicle density $\rho(x, t)$. Until now, most related

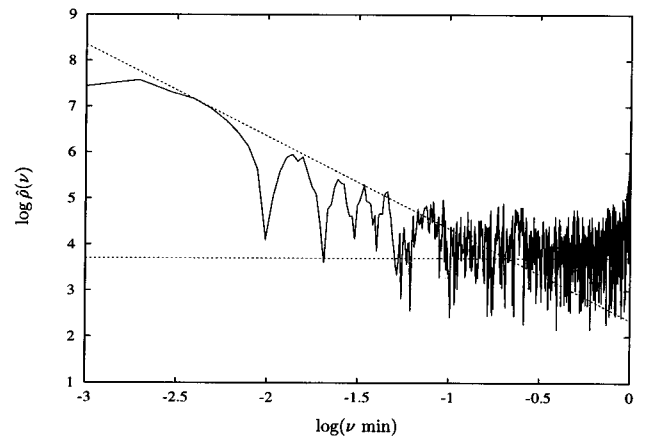


FIG. 7. The power spectrum of the time-dependent vehicle density $\rho(x, t)$ follows a power law with exponent $\delta=2.0$ at very small frequencies ν , but it is flat over large parts of the frequency range, corresponding to a *white noise*.

studies have presented theoretical or simulation results. It has been claimed that the power spectrum $\hat{\rho}(x, \nu)$ of the density $\rho(x, t)$ obeys a *power law*

$$\hat{\rho}(x, \nu) \propto \nu^{-\delta}, \quad \text{i.e.,} \quad \log \hat{\rho}(x, \nu) = C - \delta \log \nu. \quad (12)$$

For δ , the values 1.4 [14], 1.0 [15], or 1.8 [16] have been found. The empirical results in Fig. 7 indicate that the exponent δ is 2.0 at small frequencies ν , otherwise 0.0. Taking into account the logarithmic frequency scale, we can conclude that the power spectrum is flat for the most part of the frequency range. This corresponds to a *white noise*. Analogous results are found for the power spectrum of the average velocity $V(x, t)$ [1].

In summary, we found that the velocity distribution is approximately Gaussian distributed and that its skewness is negligible. We were able to reconstruct the velocity-density relation $V_e(\rho)$ and the variance-density relation $\Theta_e(\rho)$ by means of theoretical results. This allowed the determination of some density-dependent model parameters. The fluctuations of the vehicle density could be approximated by a white noise, although a power law with exponent 2.0 was found at small frequencies. All these results are necessary for realistic traffic simulations.

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